INVESTIGATION OF THE STABILITY OF THE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS AND WITH STATIONARY DELAYS IN THE ARGUMENT BY THE METHOD OF HILL

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In this article it is shown that Hill's method [1] can be applied to the investigation of the solutions of a linear differential equation with periodic coefficients and with stationary lags in the argument. The presentation is made with the aid of a second order differential equation with concentrated lags. The presented method can be extended quite easily to systems of m equations of the nth order with concentrated and uniformly distributed stationary lags in the argument.

1. The following equation is considered

$$\frac{d^2y(t)}{dt^2} + \sum_{k=0}^{s}\sum_{q=-l}^{l} a_{kq}e^{-iqt}y(t-\tau_k) = 0 \tag{1.1}$$

Here the a_{kq} are complex numbers, the τ_k are real numbers such that

$$0 = \tau_0 < \tau_1 < \ldots \tau_s \leqslant h,$$

and l is a positive number. The problem is to find, for positive t, a solution $y(t)$ that satisfies the initial conditions

$$y(t) = \varphi(t) \quad (h \leqslant t < 0), \qquad y(0) = y_0^{(0)}, \quad \frac{dy}{dt}(0) = y_0^{(1)} \tag{1.2}$$

The function $\varphi(t)$ is absolutely integrable over $h \leqslant t < 0$.

Let $f(p)$ be the Laplace transform [2] of the desired solution of Equation (1.1) satisfying the initial conditions (1.2).

Multiplying (1.1) by e^{-pt} and integrating with respect to t between the limits 0 and $+\infty$, we obtain the difference equation for the determination of $f(p)$:

$$p^2 f(p) + \sum_{q=-l}^{l} b_q(p+qi)\, f(p+qi) = \psi(p) \tag{1.3}$$

Here

$$b_q(p) = \sum_{k=0}^{s} a_{kq}\, e^{-\tau_k p}, \qquad \psi(p) = p y_0^{(0)} + y_0^{(1)} - \sum_{q=-l}^{l} \psi_q(p+qi) \tag{1.4}$$

$$\psi_q(p) = \sum_{k=1}^{s} a_{kq} \int_{-\tau_k}^{0} \psi(\tau)\, e^{-p(\tau+\tau_k)}\, d\tau \tag{1.5}$$

The functions $b_q(p)$ in (1.4) are bounded in the half-plane Re $p \geqslant c =$ const. Replacing p in (1.3) by $(p + ki)$ and dividing the obtained difference equation by $-k^2$ $(k = \pm 1, \pm 2, \pm 3, \ldots)$, we obtain an infinite system of linear algebraic equations in the unknowns $f(p + ki)$:

$$-k^{-2} f(p+ki) - \sum_{q=-l}^{l} k^{-2} b_q(p+(k+q)i)\, f(p+(k+q)i) = -k^{-2}\psi(p+ki) \tag{1.6}$$

$$(k = \pm 1, \pm 2, \pm 3, \ldots)$$

The complex variable p in (1.6) and (1.3) will be treated as a parameter. Solving the system of Equations (1.3) and (1.6) by means of Cramer's formula, we obtain

$$f(p) = \frac{1}{\Delta(p)} \begin{vmatrix} \cdot & \cdots & \cdots & \cdots & \cdot \\ \cdot & -[(p+i)^2 + b_0(p+i)] & -\psi(p+i) & -b_{-2}(p-i) & \cdot \\ \cdot & b_1(p+i) & \psi(p) & b_{-1}(p-i) & \cdot \\ \cdot & -b_2(p+i) & -\psi(p-i) & -[(p-i)^2 + b_0(p-i)] & \cdot \\ \cdot & \cdots & \cdots & \cdots & \cdot \end{vmatrix} \tag{1.7}$$

Here, $\Delta(p)$ denotes the infinite determinant of the system (1.3), (1.6).

$$\Delta(p) = \begin{vmatrix} \cdot & \cdots & \cdots & \cdots & \cdot \\ \cdot & -[(p+i)^2 + b_0(p+i)] & -b_{-1}(p) & -b_{-2}(p-i) & \cdot \\ \cdot & b_1(p+i) & p^2 + b_0(p) & b_{-1}(p-i) & \cdot \\ \cdot & -b_2(p+i) & -b_1(p) & -[(p-i)^2 + b_0(p-i)] & \cdot \\ \cdot & \cdots & \cdots & \cdots & \cdot \end{vmatrix} \tag{1.8}$$

One can show that the determinant $\Delta(p)$ in (1.8) converges absolutely and uniformly [3] in every bounded region Σ of the complex plane p. The product of the diagonal elements $A(p)$ of the determinant $\Delta(p)$ can be represented in the form

$$A(p) = [p^2 + b_0(p)] \prod_{k=1}^{\infty} \left(1 - \frac{2ip}{k} - \frac{p^2 + b_0(p+ki)}{k^2}\right)\left(1 + \frac{2ip}{k} - \frac{p^2 + b_0(p-ki)}{k^2}\right) \quad (1.9)$$

The sum of all the nondiagonal elements of the determinant $\Delta(p)$ in (1.8) is dominated by the convergent series

$$\left| \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \sum_{\substack{q=-l \\ q\neq 0}}^{l} -k^{-2} b_q(p+(k+q)i) \right| \leqslant 2 \sum_{k=1}^{\infty} k^{-2} \sum_{q=-l}^{l} \left| a_{kq} \left| \max_{p\in\Sigma} \right| e^{-\tau_k p} \right| \quad (1.10)$$

Formulas (1.9) and (1.10) imply the absolute convergence of the determinant $\Delta(p)$ of (1.8) if $p \in \Sigma$. If in (1.7) and (1.8) we take a determinant of finite order, then we obtain an approximate solution $f(p)$. Its original function will be taken as an approximate solution of Equation (1.1) with the conditions (1.2).

2. Let us consider the analytic properties of the determinant $\Delta(p)$ in (1.8). From what has been said it follows that $\Delta(p)$ is an entire function of p, and also of the parameters a_{kq}, τ_k (1.1). The center element $c(p)$ of the determinant $\Delta(p)$

$$c(p) = p^2 + \sum_{k=0}^{s} a_{k0}\, e^{-\tau_k p} \quad (2.1)$$

is an entire function of p. This function has no zeros when Re $p \geqslant \alpha$, if α is sufficiently large. The product of the equation of the diagonal elements $A(p)$ in (1.9) is a periodic function of p of period i because

$$A(p+i) = c(p+i) \lim_{r\to\infty} \prod_{\substack{k=-r \\ k\neq 0}}^{s} [-k^{-2} c(p+(k+1)i] = c(p) \lim_{r\to\infty} \prod_{\substack{k=-r \\ k\neq 0}}^{r} [-k^{-2} c(p+ + ki)[\lim \frac{c(p+(r+1)i)}{c(p-ri)} = A(p) \quad (2.2)$$

We shall make use of the notation

$$c_q(p) = b_q(p+qi)[p^2 + b_0(p)]^{-1} \quad (2.3)$$

If we move the diagonal element of each row of the determinant (1.8) behind the symbol for the determinant, we obtain

$$\Delta(p) = D(p)\, A(p) \quad (2.4)$$

where $D(p)$ is a new convergent determinant

$$D(p) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & c_{-1}(p+i) & c_{-2}(p+i) & \cdot \\ \cdot & c_1(p) & 1 & c_{-1}(p) & \cdot \\ \cdot & c_2(p-i) & c_1(p-i) & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \quad (2.5)$$

It is obvious that the determinant $D(p)$ is periodic of period i.

Hence, we have proved the following theorem.

Theorem 2.1. Hill's determinant $\Delta(p)$ in (1.8), constructed for the differential equation (1.1) with periodic coefficients and stationary lag of the argument, is an entire periodic function of p with period i.

From (2.5), (2.3) and (1.4) it follows that $D(p) \to 1$ when $\operatorname{Re} p \to +\infty$. Since $b_0(p) \to a_{00}$ in (1.4) when $\operatorname{Re} p \to +\infty$, we obtain, by retaining the term with largest absolute value, the asymptotic expression for $A(p)$ of (1.9) when $\operatorname{Re} p \to +\infty$,

$$A(p) \sim (p^2 + a_{00}) \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} [-k^{-2}((p+ki)^2 + a_{00})] = \frac{1}{2\pi^2}(\cosh 2\pi p - \cosh 2\pi \sqrt{-a_{00}}) \sim \frac{e^{2\pi p}}{4\pi^2} \tag{2.6}$$

In the particular case when the lags τ_k in (1.1) are multiples of 2π, the function $b_0(p)$ in (1.4) will be periodic with period i, and we obtain, when $\operatorname{Re} p \to +\infty$, the equation

$$A(p) = \frac{1}{2\pi^2}\left[\cosh 2\pi p - \cosh 2\pi \left(-\sum_{k=0}^{s} a_{k0} \exp\{-\tau_k p\}\right)^{\frac{1}{2}}\right] = \frac{e^{2\pi p}}{4\pi^2} + O(1) \tag{2.7}$$

Let us make the following substitution in (1.8)

$$\rho = \exp\{-2\pi p\} \tag{2.8}$$

Because of the periodicity of the determinant $\Delta(p)$, the function

$$\Phi(\rho) = \rho\Delta\left(-\frac{1}{2\pi}\ln\rho\right)4\pi^2 = 1 + O(\rho), \qquad \rho \to 0 \tag{2.9}$$

is a single-valued function without finite poles, namely, it is an entire function of ρ. By Weierstrass' theorem [2, p. 407] we have

$$\Phi(\rho) = \exp(g(\rho)) \prod_{n=1}^{\infty} \left(1 - \frac{\rho}{\rho_n}\right) \exp\left\{\frac{\rho}{\rho_n} + \frac{1}{2}\frac{\rho^2}{\rho_n^2} + \ldots + \frac{1}{k_n}\left(\frac{\rho}{\rho_n}\right)^{k_n}\right\} \tag{2.10}$$

Here $g(\rho)$ is an entire function of ρ, $g(0) = 0$, the ρ_n are the zeros of $\Phi(\rho)$ when $n \to \infty$, and the k_n are certain integers which will guarantee the convergence of (2.10). Making use of the notation $p_j = -0.5\,\pi^{-1}\ln\rho_j$, we obtain from (2.10) and (2.9) the general form of the analytic representation of $\Delta(p)$ in (1.8):

$$\Delta(p) = 0{,}25\pi^{-2}\exp\{2\pi p\}\exp[g(\exp\{-2\pi p\})] \times$$

$$\times \prod_{n=}^{\infty} (1 - \exp\{2\pi(p_j - p)\}) \exp\{2\pi(p_j - p)\} + \ldots + \frac{1}{k_n}\exp\{2\pi k_n(p_j - p)\}\} \tag{2.11}$$

$$g(\rho) = g_1\rho + g_2\rho^2 + g_3\rho^3 + \ldots +, \quad \lim_{n\to\infty} \sqrt[n]{|g_n|} = 0, \qquad \operatorname{Re} p_n \to -\infty \text{ when } n \to \infty \quad (2.12)$$

The determination of the behavior of the numbers p_n and g_n when $n \to \infty$ is still an unsolved problem.

3. The problem of the stability of the solutions of Equation (1.1) involves the evaluation of the characteristic exponents p_n of the solution of Equation (1.1). From theorem 2.1 it follows that the transform $f(p)$ (1.7) of the solution $y(t)$ is a meromorphic function of p with poles at the points

$$p_{nk} = p_n + ki \qquad (n = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots) \quad (3.1)$$

If we seek the original function $y(t)$ with the aid of the expansion given on p. 483 of [2], we obtain the next theorem.

Theorem 3.1. The solution $y(t)$ of Equation (1.1) with the initial conditions (1.2) can always be expanded into a series of the type

$$y(t) = \sum_{n=1}^{\infty} y_n(t), \qquad y_n(t) = \sum_{k=-\infty}^{\infty} (\beta_{nk}^{(0)} + \beta_{nk}^{(1)} t + \ldots + \beta_{nk}^{(r_n)} t^{r_n}) e^{(p_n + ki)t} \quad (3.2)$$

where $r_n + 1$ is the multiplicity of the zero p_n of the determinant $\Delta(p)$ (1.8).

If we substitute $y_n(t)$ from (3.2) into (1.1) we find that $y_n(t)$ is indeed a solution of Equation (1.1).

The equations for $\beta_{nk}^{(r)}$ will be satisfied because Equations (1.3) and (1.5) are satisfied by $f(p)$ which has poles of order $r_n + 1$ at the points p_{nk} (3.1).

Hence, $y_n(t)$ is an entire function of t, and the series for $y_n(t)$ (3.2) converges absolutely and uniformly for all finite values of t. This implies the asymptotic nature of the series (3.2). Thus we obtain the next theorem.

Theorem 3.2. The solution $y(t)$ of Equation (1.1) with the conditions (1.2) can always be expanded into an asymptotic series, with $t \to \infty$, of the form

$$y(t) = \sum_{n=1}^{\infty} e^{p_n t} (\alpha_n^{(0)}(t) + \alpha_n^{(1)}(t) t + \ldots + \alpha_n^{(r_n)}(t) t^{r_n}) \quad (3.3)$$

Here $\alpha_n^{(r)}(t + 2\pi) \equiv \alpha_n^{(r)}(t)$, $\operatorname{Re} p_n \to -\infty$ when $n \to +\infty$, and $r_n + 1$ stands for the multiplicity of the zero p_n of the determinant $\Delta(p)$ (1.8).

We may assume, without loss of generality, that Re $p_1 \geqslant$ Re $p_2 \geqslant$ Re $p_3 \geqslant \ldots$ Then we have the following result if Re $p^* <$ Re p_{k+1}:

$$\lim_{t\to\infty}\left| y(t) - \sum_{n=1}^{k} y_n(t) \right| \exp\{p^* t\} = 0 \tag{3.4}$$

Theorem 3.2 permits us to draw certain conclusions about the stability of the solutions of Equation (1.1) if we know the zeros p_n of the determinant Δp. This theorem can be extended to systems of linear differential equations with periodic coefficients and with stationary lags of the argument, see for example [3]. In order to find the characteristic exponents p_n, one can make use of the conditions for the existence of the solution $y(t)$ of Equation (1.1) in the form

$$y(t) = e^{pt} \sum_{k=-\infty}^{\infty} \beta_k e^{ikt} \tag{3.5}$$

4. We shall consider the Mathieu equation

$$\frac{d^2 y(t)}{dt^2} + \lambda y(t) + 2\mu y(t-\tau)\cos 2t = 0 \tag{4.1}$$

Here λ, $\mu \geqslant 0$, and $\tau \geqslant 0$ are real parameters. Equation (1.3) takes on the form

$$(p^2+\lambda) f(p) + \mu e^{-(p+2i)\tau} f(p+2i) + \mu e^{-(p-2i)\tau} f(p-2i) = \psi(p) \tag{4.2}$$

The solution of a difference equation of the type (4.2) is given in [4, p.983].

From the determinant $\Delta(p)$ one can obtain the equation [4]

$$f_0(p) - s(p) - h(p) = 0 \tag{4.3}$$

where the notation of [4, p.984] is used:

$$f_0(p) = p^2+\lambda, \qquad f_1(p) = f_{-1}(p) = \mu e^{-p\tau}, \qquad \omega = 2i \tag{4.4}$$

$$s(p) = \cfrac{f_1(p+\omega) f_{-1}(p)}{f_0(p+\omega) - \cfrac{f_1(p+2\omega) f_{-1}(p+\omega)}{f_0(p+2\omega) - \ldots}}, \quad h(p) = \cfrac{f_{-1}(p-\omega) f_1(p)}{f_0(p+\omega) - \cfrac{f_{-1}(p-2\omega) f_1(p-\omega)}{f_0(p-2\omega) - \ldots}} \tag{4.5}$$

For Equation (4.1) with $\lambda \neq k^2$ ($k = 0, 1, 2, \ldots$) Equation (4.3) takes on the form

$$p^2 + \lambda - \frac{\mu^2 e^{-2\tau(p+i)}}{(p+2i)^2+\lambda} - \frac{\mu^2 e^{-2\tau(p-i)}}{(p-2i)^2+\lambda} + O(\mu^4) = 0 \tag{4.6}$$

From Equation (4.6) we find the characteristic exponent p, which is near $i\sqrt{\lambda}$ for small values of μ

$$p = i\sqrt{\lambda} + i\frac{\mu^2}{4\sqrt{\lambda}(1-\lambda)}(\cos 2\tau\sqrt{\lambda}\cos 2\tau + \sqrt{\lambda}\sin 2\tau \sin 2\tau\sqrt{\lambda}) + \\ + \frac{\mu^2}{4\sqrt{\lambda}(1-\lambda)}(\sin 2\tau\sqrt{\lambda}\cos 2\tau - \sqrt{\lambda}\cos 2\tau\sqrt{\lambda}\sin 2\tau) + O(\mu^4) \qquad (4.7)$$

If the lag $\tau > 0$ is sufficiently small, then

$$\operatorname{Re} p = -\frac{2}{3}\mu^2\tau^3 + O(|\mu^2\tau^5| + |\mu^4|) \qquad (4.8)$$

The solutions (4.1) will be asymptotically stable for small enough values $\mu > 0$, $\tau > 0$, and $\lambda \neq k^2$ ($k = 0, 1, 2, \ldots$). Suppose that $\lambda = 0.25$. Then (4.7) yields

$$\operatorname{Re} p = -\frac{2}{3}\mu^2 \sin^3\tau + O(\mu^4) \qquad (4.9)$$

For large values of the lag $\tau > 0$, $(2n+1)\pi < \tau < (2n+2)\pi$ ($n = 0, 1, 2, \ldots$), and for sufficiently small values of $\mu > 0$, the solutions of Equation (4.1) are unstable.

5. For the investigation of the resonance $\lambda = k^2$ ($k = 0, 1, 2, \ldots$) in Equation (4.1), we shall make use of the following lemma.

Lemma 5.1. Let $\varphi(p, \mu)$ be a holomorphic function of μ and p when $|\mu| < \varepsilon$ and $|p| < \varepsilon$. Let us consider the equation

$$\varphi(p, \mu) \equiv a_0(\mu) + a_1(\mu)p + a_2(\mu)p^2 + a_3(\mu)p^3 + \ldots = 0 \qquad (5.1)$$
$$O(a_0) = O(\mu^2), \quad O(a_1) = (\mu), \quad O(a_n) = O(1) \quad (n = 2, 3, \ldots), \quad a_2(\mu) > 0$$

If it is known that two of the smallest (in absolute value) roots p_1, p_2 of Equation (5.1) are conjugates of each other, then a necessary and sufficient condition for the negativeness of the Re p_1 and Re p_2 is given by

$$\varphi(0, \mu) = a_0(\mu) > 0 \qquad (5.2)$$

$$a_1 - \frac{a_0a_2a_3^{\,2}}{a_2^2 - a_1a_3} + \frac{a_0a_2a_4(a_1a_2 - a_0a_3)}{(a_2^2 - a_1a_3)^2} + O(\mu^4) > 0 \qquad (5.3)$$

The proof of this lemma can be obtained from Weierstrass's theorem [5, p.9] by dividing $\varphi(p, \mu)$ of (5.1) by a factor, a quadratic function of p, and with the use of Hurwitz's condition [2, p.427].

Example 5.1. We shall determine the conditions for the stability of the solutions of Equation (4.1) when $\mu \approx 0$, $\lambda \approx 0$. Applying the Lemma 5.1 to Equation (4.6) and taking into account the terms of order less than $O(\mu^6 + \mu^4|\lambda| + \lambda^2\mu^2)$, we obtain the conditions for stability when

$|\mu|$ and $|\lambda|$ are small:

$$\lambda + 0.5\,\mu^2 \cos 2\tau + 0.125\,\mu^2\lambda \cos 2\tau + \frac{\mu^4}{128}\cos 8\tau > 0 \tag{5.4}$$

$$\mu^2\left[\left(-\tau - \tau\lambda + \frac{2}{3}\tau^3\lambda\right)\cos 2\tau + \left(\frac{1}{2} + \frac{\lambda}{2} - \lambda\tau^2\right)\sin 2\tau\right] + \mu^4\left[\left(-\frac{\tau}{4} - \frac{\tau^3}{3}\right) + \left(\frac{\tau}{4} - \frac{\tau^3}{3}\right)\cos 4\tau + \left(-\frac{1}{32} + \frac{\tau^2}{2}\right)\sin 4\tau - \frac{\tau}{32}\cos 8\tau + \frac{5}{256}\sin 8\tau\right] > 0 \tag{5.5}$$

When $\tau = 0$, the condition (5.4) reduces to Mathieu's criterion for stability. The Condition (5.5) is the second nonobvious criterion for stability. When $\tau > 0$ is small, the latter criterion takes on the form

$$\frac{4}{3}\mu^2\tau^3 + O(|\mu^2\tau^5| + |\mu^2\lambda| + |\mu^4|) > 0 \tag{5.6}$$

Example 5.2. We shall find the conditions for stability of the solutions of Equation (4.1) when $|\lambda - 1|$, and $|\mu|$ are small.

Let us rewrite the condition (4.3) in a more convenient form ($2i = \omega$)

$$[f_0(p) - s(p)]\,[f_0(p - 2i) - h(p - 2i)] = f_{-1}(p - 2i)\, f_1(p) \tag{5.7}$$

After the substitution of (4.4) into Equation (5.7), the latter takes the form

$$\left[p^2 + \lambda - \frac{\mu^2 e^{-2\tau(p+i)}}{(p+2i)^2 + \lambda} - \frac{\mu^4 e^{-4\tau(p+2i)}}{[(p+2i)^2+\lambda]^2\,[(p+4i)^2+\lambda]} + O(\mu^6)\right]\left[(p-2i)^2 + \lambda - \frac{\mu^2 e^{-2\tau(p-3i)}}{(p-4i)^2+\lambda} - \frac{\mu^4 e^{-4\tau(p+4i)}}{[(p-4i)^2+\lambda]^2\,[(p-6i)^2+\lambda]} + O(\mu^6)\right] = \mu^2 e^{-2\tau(p-i)} \tag{5.8}$$

Let us set $p = i + z$ in (5.8). Expanding (5.8) in powers of z, and making use of Lemma 5.1, we obtain the inequalities

$$\left(\lambda - 1 + \frac{\mu^2\cos 4\tau}{9-\lambda} + \frac{\mu^4\cos 12\tau}{1536}\right)^2 + \frac{\mu^4\sin^2 4\tau}{64} > \mu^2 + O(\mu^6 + \mu^4|\lambda - 1|) \tag{5.9}$$

$$\frac{4}{3}\mu^2\tau^3 + O(\mu^2(\lambda-1)^2 + \mu^2\tau^5 + \mu^4) > 0 \tag{5.10}$$

Example 5.3. Let us investigate the stability of the Mathieu equation with lag and friction

$$\frac{d^2y(t)}{dt^2} + \mu c\frac{dy(t)}{dt} + \lambda y(t) + 2\mu\cos 2t\, y(t-\tau) = 0, \qquad c > 0 \tag{5.11}$$

when $|\lambda - 1|$, $|\mu|$ are sufficiently small. For the purpose of finding the characteristic exponents it is advisable to use (5.7), where one has to set

$$f_0(p) = p^2 + \mu cp + \lambda, \qquad f_1(p) = f_{-1}(p) = \mu e^{-p\tau}, \qquad \omega = 2i \tag{5.12}$$

Equation (5.7) now takes the form

$$\left[p^2 + \mu cp + \lambda - \frac{\lambda^2 e^{-2\tau(p+i)}}{(p+2i)^2 + \mu c(p+2i) + \lambda} + O(\mu^4)\right]\left[(p-2i)^2 + \mu c(p-2i) + \lambda - \right.$$
$$\left. - \frac{\mu^2 e^{-2\tau(p-3i)}}{(p+4i)^2 + \mu c(p+4i) + \lambda} + O(\mu^4)\right] = \mu^2 e^{-2\tau(p-i)} \tag{5.13}$$

Expanding Equation (5.13) in powers of $z = p - i$, and applying Lemma 5.1, we obtain the following conditions for stability:

$$\left(\lambda - 1 + \frac{\mu^2 \cos 4\tau}{8}\right)^2 + \mu^2\left(c - \frac{\mu}{8}\sin 4\tau\right)^2 > \mu^2 + O(\mu^4 + \mu^3|\lambda - 1|) \tag{5.14}$$

$$\mu c + \frac{4}{3}\mu^2\tau^3 + O(\mu^3 + \mu^2\tau^5 + \mu^2(\lambda-1)^2 > 0 \tag{5.15}$$

6. In the determination of the characteristic exponents it is convenient to transform the infinite determinant of Hill (1.8), (2.5) into a determinant of finite order, as is done in [6]. Let us consider the differential equation

$$\frac{d^2y(t)}{dt^2} + \lambda y(t) + 2\mu\sum_{k=1}^{\infty} a_k \cos kty(t - \tau_k) = 0, \qquad \sum_{k=1}^{\infty}|a_k| < \infty, \qquad h < \tau_k \leqslant 0 \tag{6.1}$$

The difference equation for $f(p)$ (1.3) has the form

$$(p^2 + \lambda) f(p) + \mu\sum_{k=1}^{\infty} a_k (e^{-\tau_k(p+ki)} f(p+ki) + e^{-\tau_k(p-ki)} f(p-ki) = \psi(p) \tag{6.2}$$

Suppose that $|\lambda| \ll 1$, $|\mu| \ll 1$. When $\mu = 0$, the poles of $f(p)$ are at $\pm\sqrt{-\lambda}$. Hence, one can look for the zeros of the determinant $\Delta(p)$ (1.8) in the region Σ

$$|\lambda| < \varepsilon, \quad |\mu| < \varepsilon, \quad |p| < \varepsilon \tag{6.3}$$

Let us transfer the diagonal element in each row of the determinant $\Delta(p)$ behind the symbol of the determinant, except for the one in the central row. For small $\varepsilon > 0$, the diagonal elements $-k^{-2}[(\rho + ki)^2 + \lambda]$ $(k \neq 0)$ have no zeros in the region (6.3). Therefore, the remaining determinant Det $D_1(p)$ of the matrix $D_1(p)$ converges in the region (6.3). Hence we have

$$D_1(p) = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & c_{-2}(p+i) & c_{-2}(p+i) \\ \cdot & \mu a_1 \exp\{-\tau_1(p+i) & p^2+\lambda & \mu a_1 \exp\{-\tau_1(p-i)\} \\ \cdot & c_2(p-i) & c_1(p-i) & 1 \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\| \tag{6.4}$$

$$c_k(p) = \mu(p^2+\lambda)^{-1} a_{|k|} \exp\{-\tau_k(p+ki)\} \tag{6.5}$$

When $\varepsilon > 0$ is small, the zeros of the determinants $\Delta(p)$ (1.8) and Det $D_1(p)$ coincide in (6.3). Let us consider the auxiliary infinite matrix $R(p)$ with the determinant in (6.3) not equal to zero

$$R(p) = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & c_{-1}(p+i) & c_{-2}(p+i) \\ \cdot & 0 & 1 & 0 \\ \cdot & c_2(p-i) & c_1(p-i) & 1 \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\| \underset{p\in\Sigma}{\operatorname{Det}} R(p) \neq 0 \tag{6.6}$$

The matrix $R(p)$ coincides with the matrix $D_1(p)$ (6.4) except for the center row, where all the elements are replaced by zeros while the diagonal element is replaced by one.

Therefore, in the matrix $D_1(p)\ R^{-1}(p)$ there will be (except for the center row) ones along the diagonal and zeros off the diagonal. Det $(D_1(p)\ R^{-1}(p))$ reduces to a scalar function of p

$$\operatorname{Det}(D_1(p)\,R^{-1}(p)) = \operatorname{Det} D_1(p) \operatorname{Det} R^{-1}(p), \quad \operatorname{Det} R^{-1}(p) \neq 0 \;\; p \in \Sigma \tag{6.7}$$

Let us find the matrix $(E + C(p))^{-1}$, where

$$E = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\| \qquad C(p) = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & c_{-1}(p+i) & c_{-2}(p+i) & \cdot \\ \cdot & c_1(p) & 0 & c_{-1}(p) & \cdot \\ \cdot & c_2(p-i) & c_1(p-i) & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\| \tag{6.8}$$

$$(E + C(p))^{-1} = E - C(p) + C^2(p) - C^3(p) + \dots \tag{6.9}$$

If we eliminate from the matrix $(E + C(p))^{-1}$ the elements $c_k(p)$, which can have poles in the region Σ (6.3), then we obtain $R^{-1}(p)$, and the equation $\Delta(p) = 0$ takes on the form

$$p^2 + \lambda - \mu^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{a_k a_{-k} \exp[-\tau_k(p+ki) - \tau_{-k}p]}{(p+ki)^2+\lambda} +$$

$$+ \mu^3 \sum_{\substack{k,\,\alpha=-\infty \\ k,\,\alpha\neq 0}}^{\infty} \frac{a_k a_{\alpha-k} a_{-\alpha} \exp\{-\tau_k(p+ki) - \tau_{\alpha-k}(p+\alpha i) - \tau_{-\alpha}p\}}{[(p+(\alpha-k)i)^2+\lambda][(p+\alpha i)^2+\lambda]} - \dots = 0 \tag{6.10}$$

It is assumed here that $\tau_k = \tau_{-k}$, $a_k = a_{-k}$. In other cases, $\lambda = 0.25k^2$ ($k = 1, 2, \ldots$), and one has to proceed in a similar manner but leave two rows unchanged, the center one and the kth one. Making use of Lemma 5.1, we obtain the condition for stability of the solutions of Equation (6.1) when $|\lambda|$ and $|\mu|$ are small:

$$\lambda + 2\mu^2 \sum_{k=1}^{\infty} \frac{a_k^2 \cos k\tau_k}{k^2 - \lambda} + O(\mu^3) > 0 \qquad (6.11)$$

$$4\mu^2 \sum_{k=1}^{\infty} a_k^2 \left(\frac{\tau_k \cos k\tau_k}{\lambda - k^2} + \frac{k \sin k\tau_k}{(\lambda - k^2)^2} \right) + O(\mu^3 + \mu^2 |\lambda|) > 0 \qquad (6.12)$$

The second of these conditions is not independent on the first one.

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